Chapter 7

Dimensional Analysis,
Similitude,
and Modeling

Introduction

HISTORICAL CONTEXT

John Smeaton (1724-1792) first used scale models for systematic experimentation.

William Froude (1810-1871) first proposed laws for estimating ship hull drag from model tests.

Aimee Vaschy, Lord Rayleigh, D. Riabouchinsky, E. Buckingham all made significant contributions to dimensional analysis and similitude.

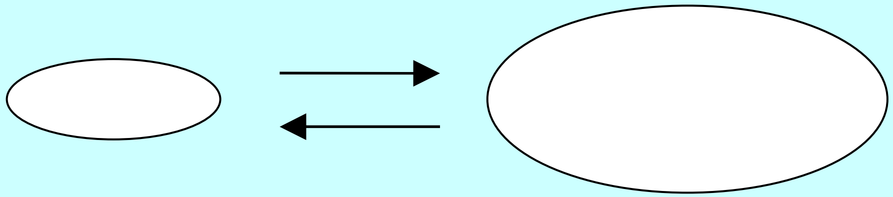
Jean B. J. Fourier (1768-1830) first formulated a theory of dimensional analysis.

Osborne Reynolds (1842-1912) first used dimensionless parameters to analyze experimental results.

Moritz Weber (1871-1951) assigned the name Reynolds number and Froude number.

Introduction

- There remain a large number of problems that rely on experimentally obtained data for their solution.
- The solutions to many problems is achieved through the use of a combination of analysis and experimental data.
- An obvious goal of any experiment is to make the results as widely applicable as possible.
- Concept of similitude



model prototype

V7.1 Real and model flies

It is necessary to establish the relationship between the laboratory model and the actual system, from which how to best conduct experiments and employ their results can be realized.

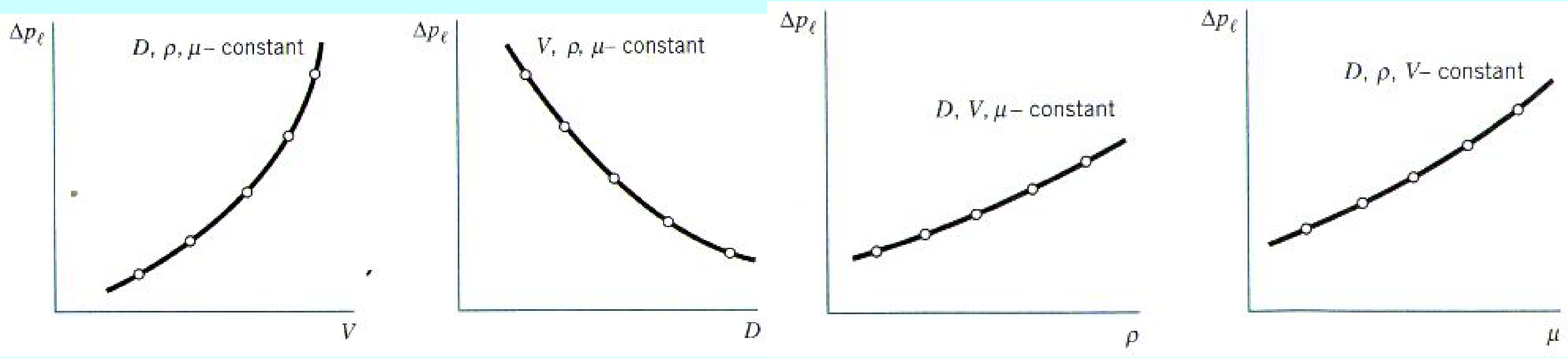
7.1 Dimensional Analysis

- Consider Newtonian fluid through a long smooth-walled, horizontal, circular pipe.

Determine pressure drop

$$\Delta p_\ell = f\left(D, \rho, \mu, V\right)$$

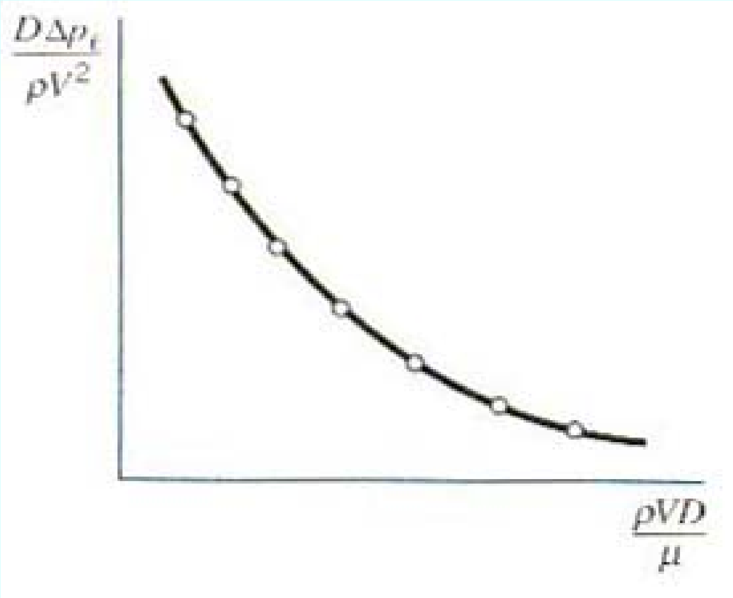
$$\Delta p_\ell = \frac{\partial p}{\partial x}$$ pressure drop per unit length



How can you do for the last two cases?

Dimensional Analysis

- Consider two non-dimensional combinations of variables



$$\frac{D\Delta p_\ell}{\rho V^2} = \phi\left(\frac{\rho VD}{\mu}\right)$$

- The results of the experiment could then be represented by a single universal curve.
- The curve would be valid for any combination of smooth walled pipe, and incompressible Newtonian fluid.

Dimensional Analysis

- To obtain this curve we could choose a pipe of convenient size and fluid that is easy to work with.
- The basis for this simplification lies in the consideration of the dimensions of the variable involved.

$$\frac{D\Delta p_\ell}{\rho V^2} = \frac{L\left(F/L^3\right)}{\left(FL^{-4}T^2\right)\left(LT^{-1}\right)^2} \doteq F^0L^0T^0$$

$$\frac{\rho VD}{\mu} \doteq \frac{\left(FL^{-4}T^2\right)\left(LT^{-1}\right)\left(L\right)}{\left(FL^{-2}T\right)} \doteq F^0L^0T^0$$

$$\Delta p_\ell = F/L^3 \qquad \rho \doteq FL^{-4}T^2$$

$$D \doteq L \qquad \mu \doteq FL^{-2}T$$

$$V \doteq LT^{-1}$$

- This type of analysis is called dimensional analysis which is based on **Buckingham pi theorem**.

7.2 Buckingham pi theorem

- How many dimensionless products are required to replace the original list of variables?

"If an equation involving *k* variables is dimensionally homogeneous, it can be reduced to a relationship among *k* – *r* independent dimensionless products, where *r* is the minimum number of reference dimensions required to describe the variables."

The dimensionless products are frequently referred to as "pi terms," and the theorem is called the Buckingham pi theorem.

$$u_1 = f\left(u_2, u_3, \cdots u_k\right)$$

$$\Pi_1 = \phi\left(\Pi_2, \Pi_3, \cdots \Pi_{k-r}\right)$$

The required number of pi terms is fewer than the number of original variables by *r*, where *r* is determined by the minimum number of reference dimensions required to describe the original list of variables.

MLT, FLT

7.3 Determination of pi terms

- method of repeating variables

1: List all the variables that are involved in the problem.

Geometry of the system (such as pipe diameter)

Fluid properties (*ρ*, *μ*)

External effects (driving pressure, *V*)

It is important that all variables be independent.

2: Express each of the variables in terms of basic dimensions.

MLT, FLT

$$F = ma = MLT^{-2}$$

$$M = \frac{F}{LT^{-2}}$$

$$\therefore \rho = ML^{-3} = FL^{-4}T^{2}$$

Determination of pi terms

3: Determine the required member of pi terms.

Buckingham pi theorem:

k variables

r reference dimensions (M, L, T, or θ)

$\Rightarrow$ $k - r$ independent dimensionless groups

4: Select a number of repeating variables, where the number required is equal to the number of reference dimensions

Notes:

1. Each repeating variable must be dimensionally independent of the others.
2. Do not choose the dependent variable (e.g., Δp) as one of the repeating variables.

Determination of pi terms

5: Form a pi form by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.

$$u_i, u_1^a, u_2^b, u_3^c$$

u_i : nonrepeating variable

u_1^a, u_2^b, u_3^c : repeating variables

6: Repeat Step 5 for each of the remaining nonrepeating variables.

7: Check all the resulting pi terms to make sure they are dimensionless.

Determination of pi terms

8: Express the final form as a relationship among the pi terms, and think about what it means

$$\Pi_1 = \phi\left(\Pi_2, \Pi_3, \cdots \Pi_{k-r}\right)$$

The actual functional relationship among the pi terms must be determined by experiments.

Determination of pi terms

- Reconsider pipe pressure drop problem

$$\Delta p_\ell = f\left(D, \rho, \mu, V\right)$$

$$\Delta p_\ell \doteq FL^{-3}, D \doteq L, \rho \doteq FL^{-4}T^2, \mu \doteq FL^{-2}T, V = LT^{-1}$$

- k=5, basic dimensions: FLT r=3, pi terms: 5-3=2

$$\Pi_1 = \Delta p_\ell D^a V^b \rho^c$$

$$\left(FL^{-3}\right)L^a\left(LT^{-1}\right)^b\left(FL^{-4}T^2\right)^c \doteq F^0L^0T^0$$

$1+c=0$	F	$a=1$
$-3+a+b-4c=0$	L	$b=-2$
$-b+2c=0$	T	$c=-1$

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2}$$

Determination of pi terms

$$\Pi_2 = \mu D^a V^b \rho^c$$

$$\left(FL^{-2}T\right)L^a\left(LT^{-1}\right)^b\left(FL^{-4}T^2\right)^c \doteq F^0L^0T^0$$

$$1 + c = 0 \qquad F \qquad a = -1$$

$$-2 + a + b - 4c = 0 \qquad L \qquad b = -1 \rightarrow \quad \Pi_2 = \frac{\mu}{DV\rho}$$

$$1 - b + 2c = 0 \qquad T \qquad c = -1$$

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} \doteq \frac{\left(FL^{-3}\right)\left(L\right)}{\left(FL^{-4}T^2\right)\left(LT^{-1}\right)^2} \doteq F^0L^0T^0$$

$$\Pi_2 = \frac{\mu}{DV\rho} \doteq \frac{FL^{-2}T}{L\left(LT^{-1}\right)\left(FL^{-4}T^2\right)} \doteq F^0L^0T^0$$

$$\therefore \frac{\Delta p_\ell}{\rho V^2} = \tilde{\phi}\left(\frac{\mu}{DV\rho}\right) \quad or \quad \phi\left(\frac{\rho VD}{\mu}\right)$$

Ex 7.1 Determine drag

V7.2 Flow past a flat plate

$$\mathscr{D} = f\left(w, h, \mu, \rho, V\right)$$

repeating variable w, V, ρ

$$\mathscr{D} \doteq MLT^{-2}$$

$$w \doteq L$$

$$h \doteq L$$

$$\mu \doteq ML^{-1}T^{-1}$$

$$\rho \doteq ML^{-3}$$

$$V \doteq LT^{-1}$$

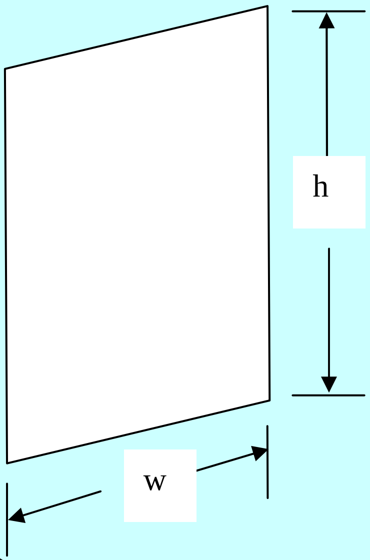
$$k - r = 6 - 3 = 3$$

$$\Pi_1 = \mathscr{D} w^a V^b \rho^c = \frac{\mathscr{D}}{w^2 V^2 \rho}$$

$$\Pi_2 = h w^a V^b \rho^c = \frac{h}{w}$$

$$\Pi_3 = \mu w^a V^b \rho^c = \frac{\mu}{w V \rho}$$

$$\frac{\mathscr{D}}{w^2 V^2 \rho} = \tilde{\phi}\left(\frac{h}{w}, \frac{\mu}{w V \rho}\right) = \phi\left(\frac{w}{h}, \frac{\rho V w}{\mu}\right)$$



7.4.1 Selection of variables

- If extraneous variables are included, then too many pi terms appear in the final solution.
- If important variables are omitted, then an incorrect result will be obtained.
- Usually, we wish to keep the problem as simple as possible, perhaps even if some accuracy is sacrificed.
- A suitable balance between simplicity and accuracy is a desirable goal.

Selection of variables

- For most engineering problems, pertinent variables can be classified into three general groups.
  Geometry: such as length, diameter, etc.
  Material properties: ρ, μ
  External Effects: V, g
- Since we wish to keep the number of variables to a minimum, it is important that all variables are independent.
- If we have a problem,

$$f\left(\rho, q, r, \cdots, u, v, w\right) = 0$$

and we know that

$$q = f_1\left(u, v, w, \cdots\right)$$

then q is not required and can be omitted. But it can be considered separately, if needed.

7.4.2 Determination of Reference Dimensions

- The use of FLT or MLT as basic dimensions is the simplest.

- Occasionally, the number of reference dimensions needed to describe all variables is smaller than the number of basic dimensions. (e.g., Ex. 7.2)

Ex. 7.2

7.4.3 Uniqueness of Pi Terms

- Consider pressure in a pipe
  Select D, V, ρ as repeating variables

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

- If instead choosing D, V, μ as repeating variables

$$\frac{\Delta p_\ell D^2}{V\mu} = \phi_1\left(\frac{\rho V D}{\mu}\right)$$

- Therefore there is not a unique set of pi terms which arises from a dimensional analysis.

Uniqueness of Pi Terms

- However, the required number of pi terms is fixed, and once a correct set is determined all other possible sets can be developed from this set by combinations of products of powers of the original set.

Thus

$$\Pi_1 = \phi\left(\Pi_2, \Pi_3\right)$$

$$\Pi_2' = \Pi_2^a \, \Pi_3^b \quad \text{where } a\,, b \text{ are arbitrary exponents}$$

Then

$$\Pi_1 = \phi_1\left(\Pi_2', \Pi_3\right)$$

$$\Pi_1 = \phi_2\left(\Pi_2, \Pi_2'\right)$$

$$\left(\frac{\Delta p_\ell \, D}{\rho V^2}\right)\left(\frac{\rho V D}{\mu}\right) = \left(\frac{\Delta p_\ell \, D^2}{V \mu}\right)$$

Uniqueness of Pi Terms

- There is no simple answer to the question:

  Which form for the pi terms is best?

- Usually, the only guideline is to keep the pi terms as simple as possible.

- Also, it may be that certain pi terms will be easier to work with in actually performing experiments.

7.6 Common Dimensionless Groups in Fluid Mechanics

TABLE 7.1

Some Common Variables and Dimensionless Groups in Fluid Mechanics

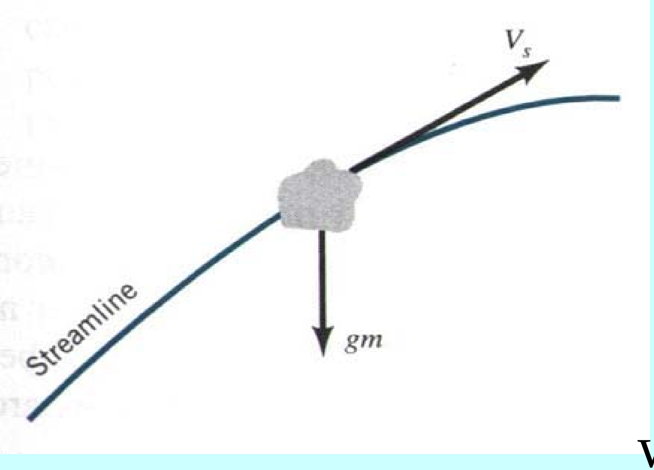
Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, ℓ; Density, ρ; Frequency of oscillating flow, ω; Pressure, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\frac{\rho V \ell}{\mu}$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\frac{V}{\sqrt{g\ell}}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{V}{c}$	Mach number,[a] Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{\omega \ell}{V}$	Strouhal number, St	$\frac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\frac{\rho V^2 \ell}{\sigma}$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

[a]The Cauchy number and the Mach number are related and either can be used as an index of the relative effects of inertia and compressibility. See accompanying discussion.

Common Dimensionless Groups in Fluid Mechanics

- Froude number, *Fr*



$$Fr = \frac{V}{\sqrt{g\ell}} = \frac{inertia\ \ force}{gravitational\ \ force}$$

$$F_I = a_s m$$

$$a_s = \frac{dV_s}{dt} = V_s \frac{dV_s}{ds}$$

where *s* is measured along the streamline

Common Dimensionless Groups in Fluid Mechanics

- If V_s and s are expressed in dimensionless form

$$V_s^* = \frac{V_s}{V}, \quad s^* = \frac{s}{\ell}$$

where V and ℓ represent some characteristic velocity and length.

$$a_s = \frac{dV_s}{dt} = V_s \frac{dV_s}{ds} = \frac{V^2}{\ell} V_s^* \frac{dV_s^*}{ds^*}$$

$$F_I = ma_s = \frac{V^2}{\ell} V_s^* \frac{dV_s^*}{ds^*} m, \quad F_G = mg$$

$$\frac{F_I}{F_G} = \frac{V^2}{g\ell} V_s^* \frac{dV_s^*}{ds^*}$$

$$Fr = \frac{V}{\sqrt{g\ell}}$$

V7.4 Froude number

For a problem in which gravity (or weight) is not important, the Froude number would not appear as an important pi term.

Common Dimensionless Groups in Fluid Mechanics

- Reynolds number

$$\mathrm{Re} = \frac{\rho V \ell}{\mu} \qquad \frac{\text{inertia force}}{\text{viscous force}}$$

V7.3 Reynolds number

Re<<1: creeping flow
Large Re: the flow can be considered nonviscous

- Euler number

$$\mathrm{Eu} = \frac{p}{\rho V^2} = \frac{\Delta p}{\rho V^2} \qquad \frac{\text{pressure force}}{\text{inertia force}}$$

Some form of the Euler number would normally be used in problems in which pressure or the pressure difference between two points is an important variable.

For problem in which cavitation is of concern,

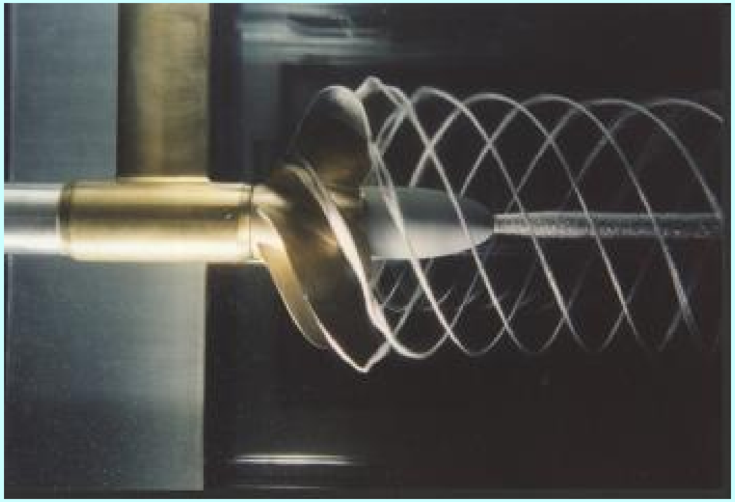
$$\frac{p_r - p_\upsilon}{\frac{1}{2}\rho V^2} \quad : \text{cavitation number}$$

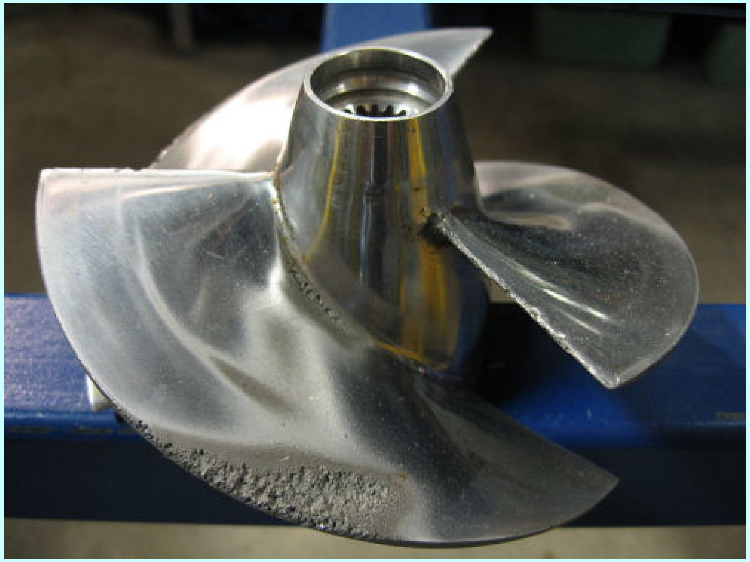
Cavitation is the formation of vapor bubbles of a flowing liquid in a region where the pressure of the liquid falls below its vapor pressure.

For example, cavitation may occur when the speed of the propeller tip is so high that the liquid pressure becomes lower than the vapor pressure

$$\frac{p_r - p_\upsilon}{\frac{1}{2}\rho V^2} \quad \text{: cavitation number}$$

www.amhrc.edu.au/images/cavtunnel-propellor2.jpeg





www.boattest.com/images-gallery/News/prop_cav.jpg

Common Dimensionless Groups in Fluid Mechanics

- **Cauchy Number** and **Mach Number**

$$\text{Ca} = \frac{\rho V^2}{E_\upsilon}, \qquad \text{Ma} = \frac{V}{c} = V\sqrt{\frac{\rho}{E_\upsilon}} \quad c = \sqrt{\frac{E_\upsilon}{\rho}}$$

$$\text{Ma}^2 = \frac{\rho V^2}{E_\upsilon} = \text{Ca}$$

When the Mach number is relatively small (less than 0.3), the inertial forces induced by the fluid motion are not sufficiently large to cause a significant change in the fluid density, and in this case the compressibility of the fluid can be neglected.

Common Dimensionless Groups in Fluid Mechanics

- Strouhal Number

$$\mathrm{St} = \frac{\omega \ell}{V}$$

important in unsteady, oscillating flow problem in which the frequency of the oscillation is ω.

It represent a measure of the ratio of inertia forces due to the unsteadiness of the flow (local acceleration, $\partial V / \partial t$) to the inertia forces due to change in velocity from point to point in the flow field (convective acceleration, $V \partial V / \partial x$).

V7.5 Strouhal number

singing wires

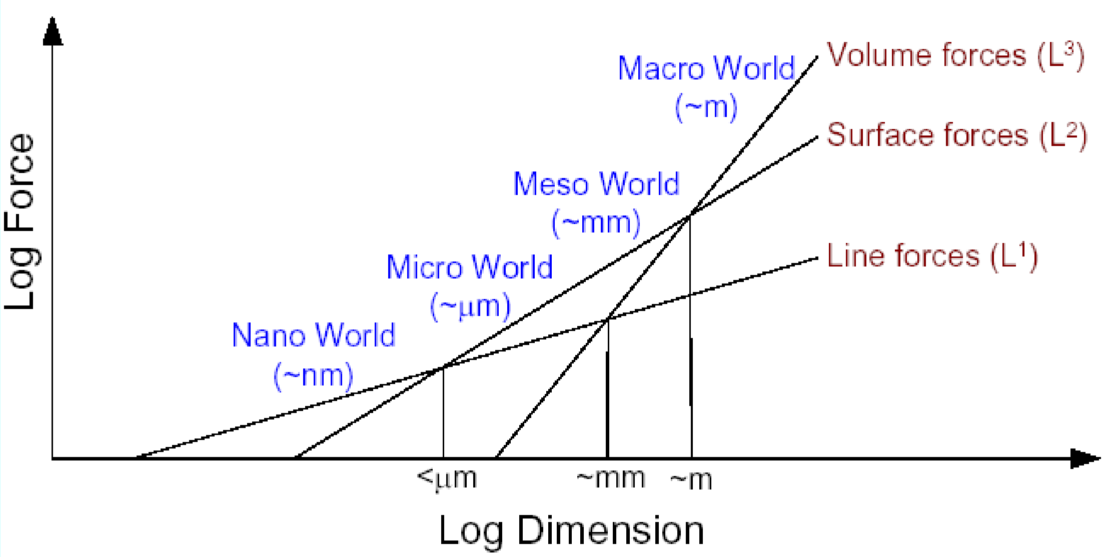
Common Dimensionless Groups in Fluid Mechanics

- Weber Number

$$\mathrm{We} = \frac{\rho V^2 \ell}{\sigma}$$

It is important when the surface tension at the interface between two fluids is significant. Surface tension is a line force (F/L), whose effects become significant, even dominant, when the scale decreases to about < 100 μm.

V7.6 Weber number



7.7 Correlation of Experimental data

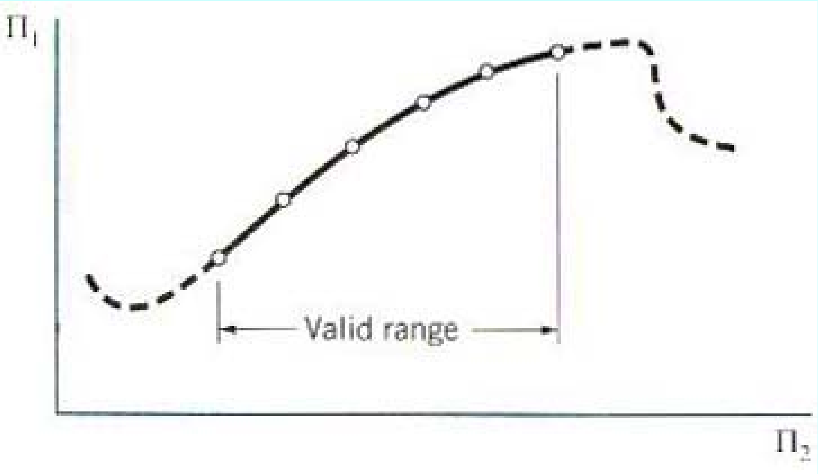
- One of the most important uses of dimensional analysis is as an aid in the efficient handling, interpretation, and correlation of experimental data.
- Dimensional analysis provide only the dimensionless groups describing the phenomenon, and not the specific relationship among the groups.
- To determine this relationship, suitable experimental data must be obtained.

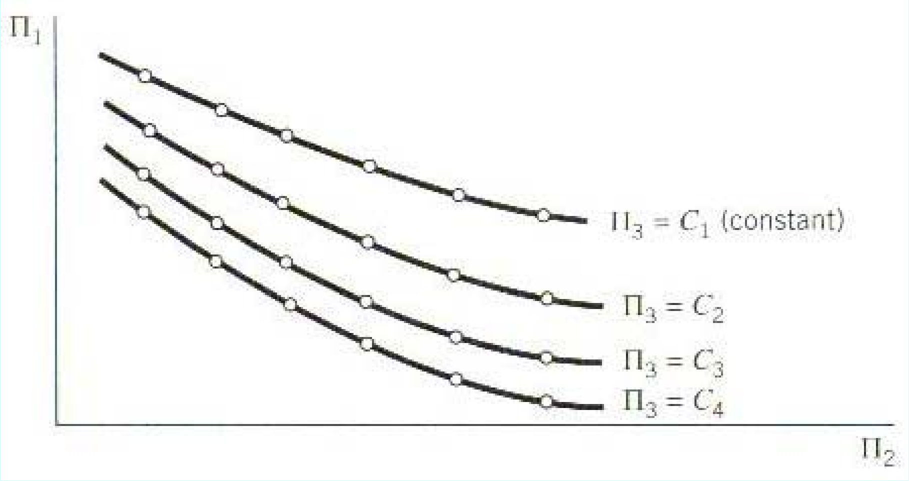
7.7.1 Problems with One Pi Term

$$\Pi_1 = C$$ where C is a constant

7.7.2 Problems with Two or More Pi Terms

$$\Pi_1 = \phi\left(\Pi_2\right) \qquad \Pi_1 = \phi\left(\Pi_2, \Pi_3\right)$$



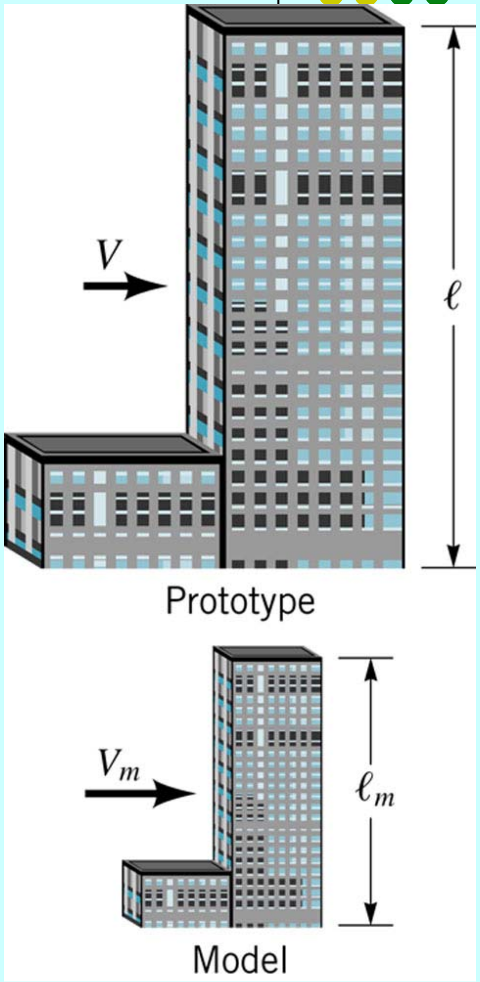


Ex. 7.3 Flow with only one Pi term V7.7 Stokes flow

Ex. 7.4 Dimensionless correlation of experimental data

7-8 Modeling and Similitude

- A model is a representation of a physical system that may be used to predict the behavior of the system in some desired respect.
- The physical system for which the predictions are to be made is called the prototype.
- Usually a model is smaller than the prototype. Occasionally, if the prototype is very small , it may be advantageous to have a model that is larger than the prototype so that it can be more easily studied.



V7.9 Environmental models

7.8.1 Theory of models

- It has been shown that

$$\Pi_1 = \phi\left(\Pi_2, \Pi_3, \cdots \Pi_n\right)$$

If the equation describes the behavior of a particular prototype, a similar relationship can be written for a model of this prototype, ie.

$$\Pi_{1m} = \phi\left(\Pi_{2m}, \Pi_{3m}, \cdots \Pi_{nm}\right)$$

where the form of the function will be the same as long as the same phenomenon is involved in both the prototype and the model.

Theory of models

- Therefore, if the model is designed and operated under the following conditions,

$$\begin{aligned}\Pi_{2m} &= \Pi_2,\\ \Pi_{3m} &= \Pi_3,\\ &\vdots\\ \Pi_{nm} &= \Pi_n\end{aligned}$$

model design condition or similarity requirement or modeling laws

then with the presumption that the form of is the same for model and prototype, it follows that

$$\Pi_1 = \Pi_{1m}$$

- prediction equation

Theory of models

$$\mathscr{D} = f\left(w, h, \mu, \rho, V\right)$$

pi theorem indicates

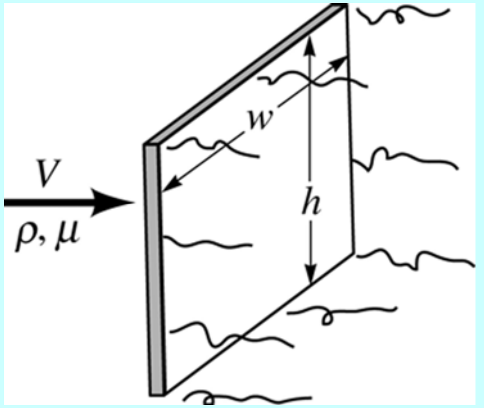
$$\frac{\mathscr{D}}{w^2 \rho V^2} = \phi\left(\frac{w}{h}, \frac{\rho V w}{\mu}\right)$$

$$\frac{\mathscr{D}_m}{w_m^2 \rho_m V_m^2} = \phi\left(\frac{w_m}{h_m}, \frac{\rho_m V_m w_m}{\mu_m}\right)$$

model design conditions

$$\frac{w_m}{h_m} = \frac{w}{h}, \quad \frac{\rho_m V_m w_m}{\mu_m} = \frac{\rho V w}{\mu}$$

$$w_m = \frac{h_m}{h} w, \quad V_m = \frac{\mu_m}{\mu} \frac{\rho}{\rho_m} \frac{w}{w_m} V$$



thus, $\dfrac{\mathscr{D}}{w^2 \rho V^2} = \dfrac{\mathscr{D}_m}{w_m^2 \rho_m V_m^2}$

or, $\mathscr{D} = \left(\dfrac{w}{w_m}\right)^2 \left(\dfrac{\rho}{\rho_m}\right)^2 \left(\dfrac{V}{V_m}\right)^2 \mathscr{D}_m$

Thus, to achieve similarity between model and prototype behavior, all the corresponding pi terms must be equated between model and prototype.

Theory of models

Similarity:

- Geometric similarity: (length scale)

  A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale rate. (including angles)

- Kinematic similarity: (length scale and time scale, ie. velocity scale)

  The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times

- Dynamic similarity

  Model and prototype have the same length-scale ratio, time-scale ratio, and force-scale (mass-scale) ratio

Ex. 7.5 V7.10 Flow past an ellipse

7.8.2 Model Scales

- The ratio of like quantities for the model and prototype naturally arises from the similarity requirements.

$$\frac{\ell_1}{\ell_2} = \frac{\ell_{1m}}{\ell_{2m}} \qquad \text{length scale}$$

$$\frac{V_1}{V_2} = \frac{V_{1m}}{V_{2m}} \qquad \text{velocity scale}$$

etc.

7.8.3 Practical Aspects of Using Models

- Validation of Model Design

  It is desirable to check the design experimentally whenever possible.

  May run tests with a series of models of different sizes.

- Distorted Model

  If one or more of the similarity requirements are not met, e.g., $\Pi_{2m} \neq \Pi_2$, then it follows that the prediction equation, $\Pi_1 = \Pi_{1m}$ is not true, i.e., $\Pi_1 \neq \Pi_{1m}$.

  Models for which one or more of the similarity requirements are not satisfied are called distorted models.

V7.12 Distorted river model

Practical Aspects of Using Models

- e.g., open channel or free surface flow

$$\mathrm{Re} = \frac{\rho V \ell}{\mu}, \quad Fr = \frac{V}{\sqrt{g\ell}}$$

Froude number similarity:

$$\frac{V_m}{\sqrt{g_m \ell_m}} = \frac{V}{\sqrt{g\ell}} \quad \rightarrow \quad \frac{V_m}{V} = \sqrt{\frac{\ell_m}{\ell}} = \sqrt{\lambda_\ell}$$

Reynolds number similarity:

$$\frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu} \quad \rightarrow \quad \frac{V_m}{V} = \frac{\mu_m}{\mu} \frac{\rho}{\rho_m} \frac{\ell}{\ell_m} = \frac{\nu_m}{\nu} \frac{\ell}{\ell_m}$$

$$\Rightarrow \quad \frac{\nu_m}{\nu} = \frac{V_m}{V} \frac{\ell_m}{\ell} = \sqrt{\frac{\ell_m}{\ell}} \frac{\ell_m}{\ell} = \left(\lambda_\ell\right)^{3/2}$$

The common fluid used is water, therefore the above requirement will not be satisfied.

7.9 Some Typical Model Studies
7.9.1 Flow Through Closed Conduits

- Example: flow through valves, fittings, metering devices.
- For low Mach numbers (Ma < 0.3), any dependent Pi term (such as pressure drop) can be expressed as,

$$\text{Dependent pi term} = \phi\left(\frac{\ell_i}{\ell}, \frac{\varepsilon}{\ell}, \frac{\rho V \ell}{\mu}\right)$$

where ε : surface roughness; ℓ : a particular length dimension;
ℓ_i , i=1,2, … : a series of length terms of the system

$$\frac{\ell_{im}}{\ell_m} = \frac{\ell_i}{\ell}, \quad \frac{\varepsilon_m}{\ell_m} = \frac{\varepsilon}{\ell}, \quad \frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu}$$

If the pressure drop is the dependent variable then,

$$\Pi_1 = \frac{\Delta p}{\rho V^2}, \quad \frac{\Delta p_m}{\rho_m V_m^2} = \frac{\Delta p}{\rho V^2}$$

Ex. 7.6

Flow Through Closed Conduits

- For large Reynolds numbers, inertial forces >>viscous forces, and in this case it may be possible to neglect viscous effects; i.e., it would not be necessary to maintain Reynolds number similarity between model and prototype.
  However, both model and prototype have to operate at large Reynolds number, and the dependent pi term ceases to be affected by changes in Re. (will be shown later)
- For flows cavitation phenomenon, then the vapor pressure p_υ becomes an important variable and an additional similarity requirement such as equality of the cavitation number is required

$(p_r - p_\upsilon) / \frac{1}{2}\rho V^2$, where p_r is some reference pressure.

7.9.2 Flow Around Immersed Bodies

- Flow around aircraft, automobiles, golf balls , and buildings.

Dependent pi term $= \phi\left(\dfrac{\ell_i}{\ell}, \dfrac{\varepsilon}{\ell}, \dfrac{\rho V \ell}{\mu}\right)$

Frequently drag $\mathscr{D}$ is of interest.

Thus $$C_D = \frac{\mathscr{D}}{\frac{1}{2}\rho V^2 \ell^2} = \phi\left(\frac{\ell_i}{\ell}, \frac{\varepsilon}{\ell}, \frac{\rho V \ell}{\mu}\right)$$

- geometric similarity: $$\frac{\ell_{im}}{\ell_m} = \frac{\ell_i}{\ell}, \quad \frac{\varepsilon_m}{\ell_m} = \frac{\varepsilon}{\ell}$$

- Reynolds number similarity: $$\frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu}$$

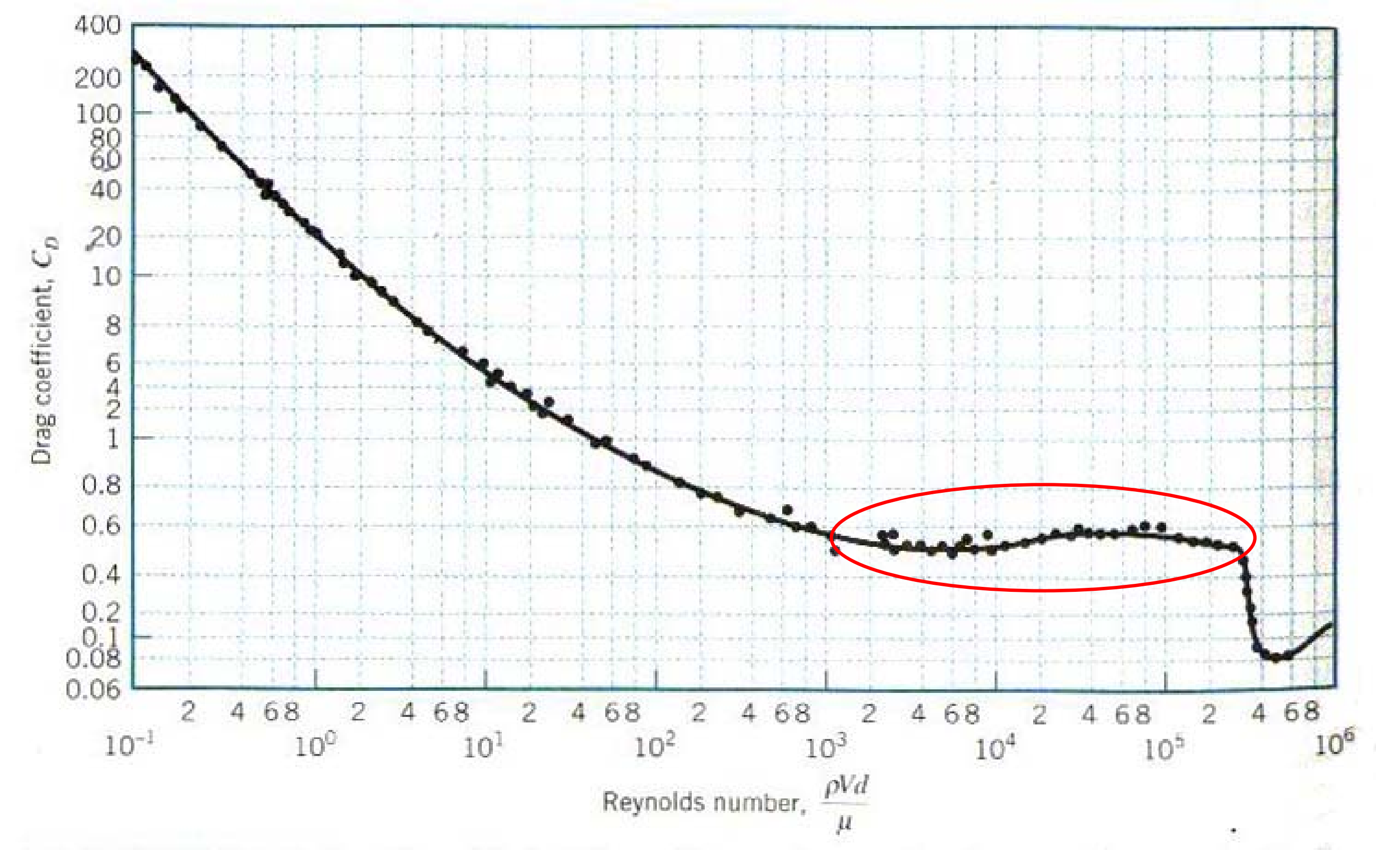
$$\rightarrow V_m = \frac{\mu_m}{\mu}\frac{\rho}{\rho_m}\frac{\ell}{\ell_m}V = \frac{\nu_m}{\nu}\overbrace{\frac{\ell}{\ell_m}}^{>1}V$$

V7.14 Model airplane test in water

To reduce $V_m \rightarrow \dfrac{\nu_m}{\nu} < 1$ ($\nu_{\text{water}}/\nu_{\text{air}} \sim 0.1$), or $\rho_m > \rho$ (increase p)

Flow Around Immersed Bodies

- Fortunately, in many situations the flow characteristics are not strongly influenced by Re over the operating range of interest. For high Re, inertial forces are dominant, and C_D is essentially independent of Re (Fig. 7.7--C_D for a sphere).



$$C_D = \frac{\mathcal{D}}{\frac{1}{2} A \rho V^2},$$

$$A = \pi d^2 / 4$$

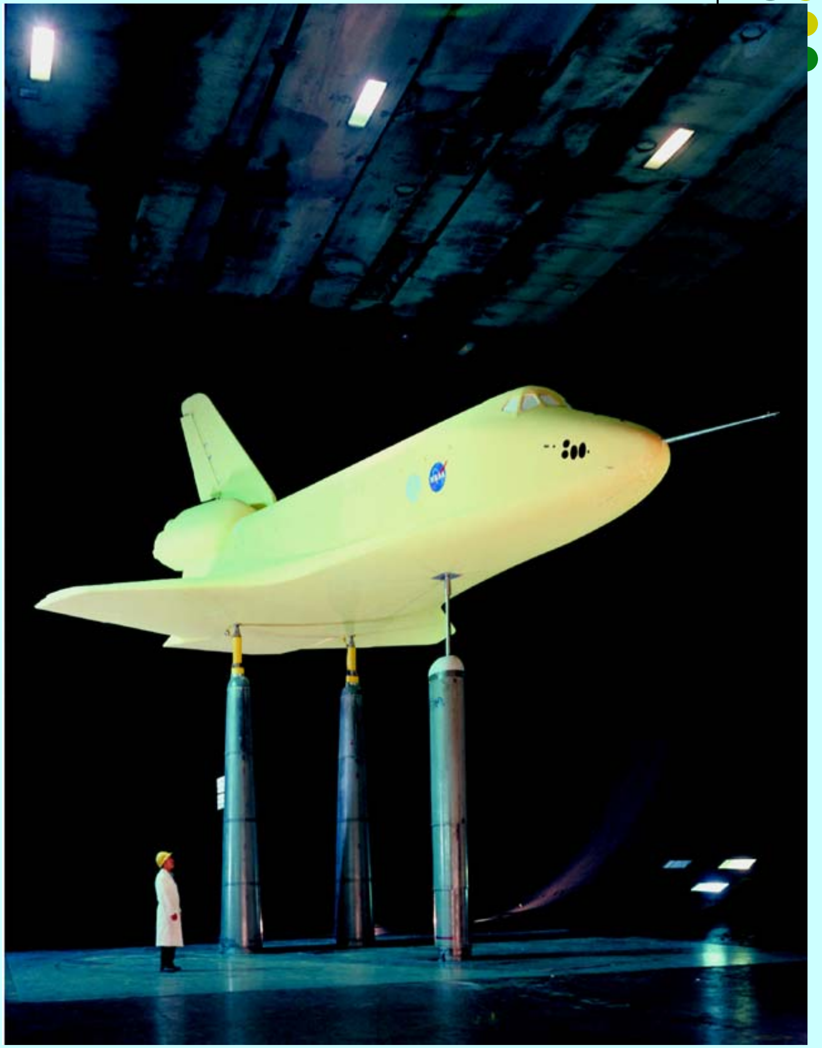
Flow Around Immersed Bodies

- NASA Ames

  40 × 80 ft 345 mil/hr

  12 × 24 m 552 km/hr

  test section



Wind-tunnel testing is an essential part of aircraft design. A 0.36-scale model of the Space Shuttle orbiter is tested in NASA Ames 40 × 80 ft wind tunnel. Photograph courtesy of NASA.

Ex. 7.7

V7.15 Large scale wind tunnel

Flow Around Immersed Bodies

- For problem of high Mach number (Ma>0.3), compressibility effect grows significant.

$$\frac{V_m}{c_m} = \frac{V}{c}$$

Combined with Reynolds number similarity

$$\rightarrow \frac{c}{c_m} = \frac{\nu}{\nu_m}\frac{\ell_m}{\ell}$$

- In high speed aerodynamics the prototype fluid is usually air,

$c = c_{\mathrm{m}}$, $\nu = \nu_{\mathrm{m}}$,

and it is difficult to satisfy the above condition, for reasonable length scales.

- Thus, models involving high speed flows are often distorted with respect to Reynolds number similarity, but Mach number similarity is maintained.

Incomplete Similarity: Automobile and Truck Tests

3/8 scale wind tunnel test for automobiles :
→240km/h wind speed to match *Re* (no compressibility effect concern)

1/8 scale wind tunnel test for trucks or buses :
→ 700km/h wind speed to match *Re* (compressibility effects arise!)

→ match of *Re* unwanted. (Incomplete similarity)

Then, how to solve this problem? Use *Re*-independence of drag coeff. above a certain *Re*.

EX. 7.5 of Fox, et al. "Introduction to Fluid Mechanics," 7th Ed., 2010.

--Incomplete Similarity: Aerodynamic Drag on a Bus

7.9.3 Flow with a Free Surface

Dependent pi term $= \phi\left(\frac{\ell_i}{\ell}, \frac{\varepsilon}{\ell}, \frac{\rho V \ell}{\mu}, \frac{V}{\sqrt{g\ell}}, \frac{\rho V^2 \ell}{\sigma}\right)$

Weber number

So that

Froude number: $\frac{V_m}{\sqrt{g_m \ell_m}} = \frac{V}{\sqrt{g\ell}}, \quad g_m = g \quad \rightarrow \frac{V_m}{V} = \sqrt{\frac{\ell_m}{\ell}} = \sqrt{\lambda_\ell}$

Reynolds number: $\frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu} \quad \rightarrow \quad \frac{\nu_m}{\nu} = \left(\frac{\ell_m}{\ell}\right)^{3/2} = \left(\lambda_\ell\right)^{3/2}$ (7.15)

Weber number: $\frac{\sigma_m / \rho_m}{\sigma / \rho} = \left(\lambda_\ell\right)^2$

- For large hydraulic structures, such as dam spillways, the Reynolds numbers are large, viscous forces are small in comparison to the forces due to gravity and inertia. Therefore, Re similarity is not maintained and model designed on the basis of Froude number (why not Re?).

V7.19 Dam model

Ex. 7.8

Incomplete Similarity: Flow with a Free Surface

V7.20 Testing of large yacht model

How to find the full-scale ship resistance from model test results? Follow the following procedure: (from Fox, et al. "Introduction to Fluid Mechanics")

Model Tests (Fig. 7.2)

(a) Measure the total drag coeff. $(C_{D,T})_m$ from model tests at corresponding Fr

(b) Calculate analytically the friction drag coeff. $(C_{D,F})_m$,

(c) Find the wave drag coeff. $(C_{D,W})_m = (C_{D,T})_m - (C_{D,F})_m$.

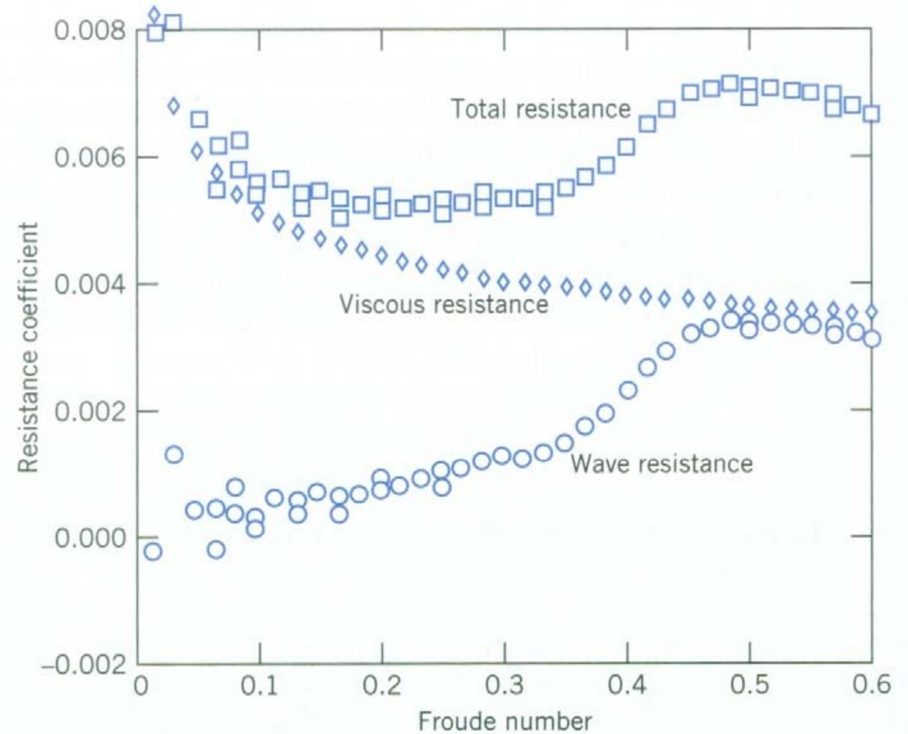


Fig. 7.2 Data from test of 1:80 scale model of U.S. Navy guided missile frigate *Oliver Hazard Perry* (FFG-7). (Data from U.S. Naval Academy Hydromechanics Laboratory, courtesy of Professor Bruce Johnson.)

Flow with a Free Surface

Prototype Predictions (Fig. 7.3)

(a) Match $(C_{D,W})_p = (C_{D,W})_m$ at corresponding Fr (Froude no. scaling),

(b) Calculate analytically $(C_{D,F})_p$,

(c) Find $(C_{D,T})_p = (C_{D,W})_p + (C_{D,F})_p$

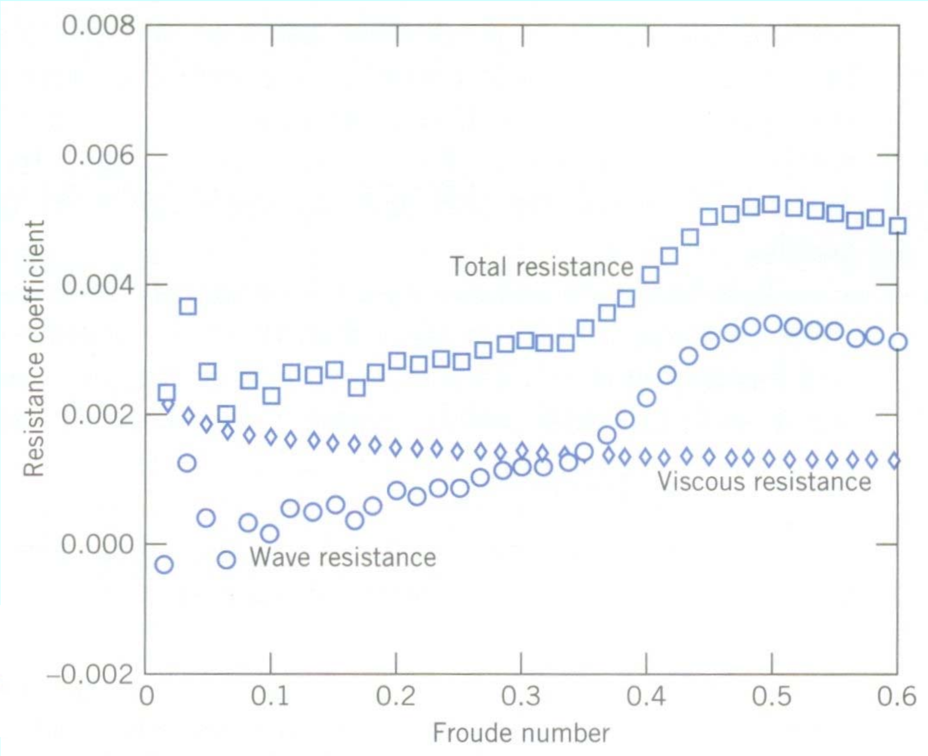


Fig. 7.3 Resistance of full-scale ship predicted from model test results. (Data from U.S. Naval Academy Hydromechanics Laboratory, courtesy of Professor Bruce Johnson.)

Note: Special treatment (adding studs) is needed for the ship model to stimulate turbulent boundary layer at proper position.

7.10 Similitude Based on Governing Differential Equations

- Consider 2-D equation

$$\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}=0$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}\right)=-\frac{\partial p}{\partial y}+\rho g+\mu\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}\right)$$

new dimensionless variables:

$$u^*=\frac{u}{V},\quad \upsilon^*=\frac{\upsilon}{V},\quad p^*=\frac{p}{p_0}$$

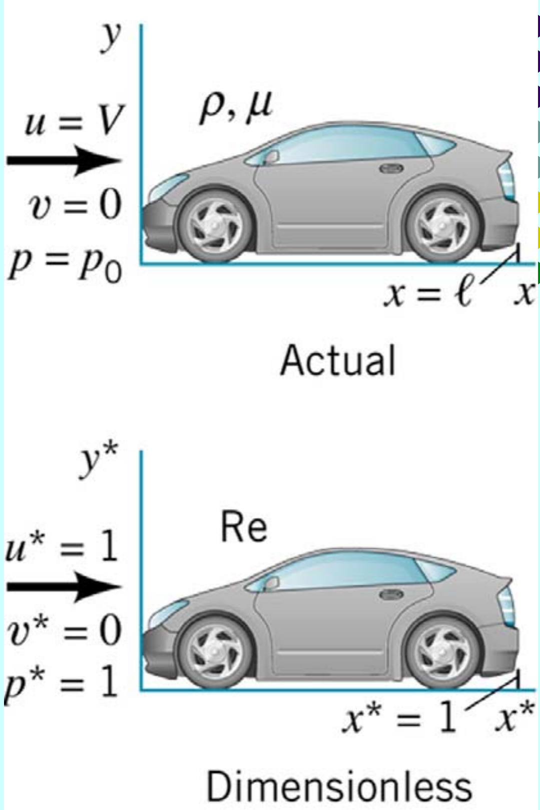
$$x^*=\frac{x}{\ell},\quad y^*=\frac{y}{\ell},\quad t^*=\frac{t}{\tau}\quad \ell:\text{reference length},\ \tau:\text{reference time}$$

- Therefore,

$$\frac{\partial u}{\partial x}=\frac{\partial Vu^*}{\partial x^*}\frac{\partial x^*}{\partial x}=\frac{V}{\ell}\frac{\partial u^*}{\partial x^*}$$

and

$$\frac{\partial^2 u}{\partial x^2}=\frac{V}{\ell}\frac{\partial}{\partial x^*}\left(\frac{\partial u^*}{\partial x^*}\right)\frac{\partial x^*}{\partial x}=\frac{V}{\ell^2}\frac{\partial^2 u^*}{\partial x^{*2}}$$



Thus

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial \upsilon^*}{\partial y^*}=0$$

$$\left[\frac{\rho V}{\tau}\right]\frac{\partial u^*}{\partial t^*}+\left[\frac{\rho V^2}{\ell}\right]\left(u^*\frac{\partial u^*}{\partial x^*}+\upsilon^*\frac{\partial u^*}{\partial y^*}\right)=-\left[\frac{p_0}{\ell}\right]\frac{\partial p^*}{\partial x^*}+\left[\frac{\mu V}{\ell^2}\right]\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\underbrace{\left[\frac{\rho V}{\tau}\right]}_{F_{I\ell}}\frac{\partial \upsilon^*}{\partial t^*}+\underbrace{\left[\frac{\rho V^2}{\ell}\right]}_{F_{Ic}}\left(u^*\frac{\partial \upsilon^*}{\partial x^*}+\upsilon^*\frac{\partial \upsilon^*}{\partial y^*}\right)=-\underbrace{\left[\frac{p_0}{\ell}\right]}_{F_P}\frac{\partial p^*}{\partial y^*}+\underbrace{\rho g}_{F_G}+\underbrace{\left[\frac{\mu V}{\ell^2}\right]}_{F_V}\left(\frac{\partial^2 \upsilon^*}{\partial x^{*2}}+\frac{\partial^2 \upsilon^*}{\partial y^{*2}}\right)$$

$F_{I\ell}$ ↑ inertia (local) force

F_{Ic} ↑ inertia (convective) force

F_P ↑ pressure force

F_G ↑ gravity force

F_V ↑ viscous force

$$\left[\frac{\ell}{\tau V}\right]\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+\upsilon^*\frac{\partial u^*}{\partial y^*}=-\left[\frac{p_0}{\rho V^2}\right]\frac{\partial p^*}{\partial x^*}+\left[\frac{\mu}{\rho V\ell}\right]\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\left[\frac{\ell}{\tau V}\right]\frac{\partial \upsilon^*}{\partial t^*}+u^*\frac{\partial \upsilon^*}{\partial x^*}+\upsilon^*\frac{\partial \upsilon^*}{\partial y^*}=-\left[\frac{p_0}{\rho V^2}\right]\frac{\partial p^*}{\partial y^*}+\left[\frac{g\ell}{V^2}\right]+\left[\frac{\mu}{\rho V\ell}\right]\left(\frac{\partial^2 \upsilon^*}{\partial x^{*2}}+\frac{\partial^2 \upsilon^*}{\partial y^{*2}}\right)$$

a form of Strouhal number

Euler number

reciprocal of Froude number square

reciprocal of Reynolds number

- If two systems are governed by these equations, then the solutions (in terms of u^*, υ^*, p^*, x^*, y^*, and t^*) will be the same if the four parameters:

$$\frac{\ell}{\tau V},\ \frac{p_0}{\rho V^2},\ \frac{V^2}{g\ell},\ \frac{\rho V\ell}{\mu}$$ are equal for the two systems.

(B.C.s must also be the same).